LABORATORIO DI MATEMATICA - ESERCIZI DI RIEPILOGO

venerdì 14 febbraio 2025 09:06

ESERCIZI SULLE DISEQUAZIONI CON VALORE ASSOLUTO:

1) $$\frac{|3-x|-2}{5x+1} < 0$$

2) $$\frac{3x-1}{2|2x-1|-3} \geq 0$$

Soluzioni:

1) $$\frac{|3-x|-2}{5x+1} < 0$$

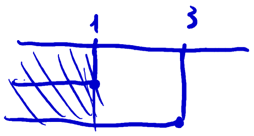
Numeratore:

$$|3-x|-2 \geq 0$$

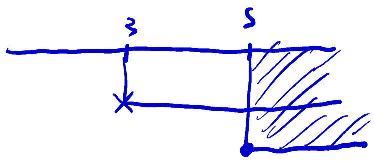
$$\begin{cases} 3-x \geq 0 \\ 3-x-2 \geq 0 \end{cases} \quad \vee \quad \begin{cases} 3-x < 0 \\ -(3-x)-2 \geq 0 \end{cases}$$

$$\begin{cases} x \leq 3 \\ x \leq 1 \end{cases} \qquad\qquad \begin{cases} x > 3 \\ -3+x-2 \geq 0 \end{cases}$$

$$\begin{cases} x > 3 \\ x \geq 5 \end{cases}$$

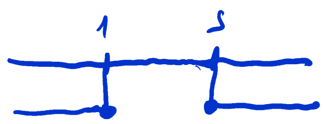


$x \leq 1$



$x \geq 5$

Numeratore: $x \leq 1 \quad \vee \quad x \geq 5$

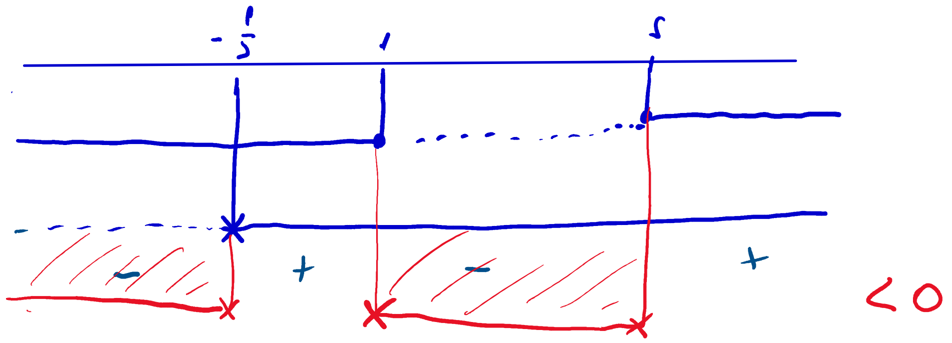


Denominatore: $5x+1 > 0$

$5x > -1$

$x > -\frac{1}{5}$

Studio del segno:



$$x < \frac{1}{5} \quad \vee \quad 1 < x < 5$$

ESERCIZIO 2

- $$\frac{3x-1}{2|2x-1|-3} \geq 0$$

Numeratore: $3x-1 \geq 0 \iff x \geq \frac{1}{3}$.

Denominatore: $2|2x-1|-3 > 0$

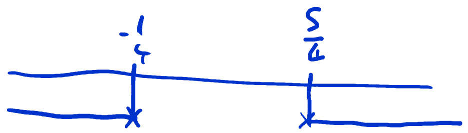
$$\begin{cases} 2x-1 \geq 0 \\ 2(2x-1)-3 > 0 \end{cases} \vee \begin{cases} 2x-1 < 0 \\ -2(2x-1)-3 > 0 \end{cases}$$

$$\begin{cases} x > \frac{1}{2} \\ 4x-2-3 > 0 \end{cases} \qquad \begin{cases} x < \frac{1}{2} \\ -4x+2-3 > 0 \end{cases}$$

$$\begin{cases} x > \frac{1}{2} \\ x > \frac{5}{4} \end{cases} \qquad \begin{cases} x < \frac{1}{2} \\ -4x-1 > 0 \end{cases}$$

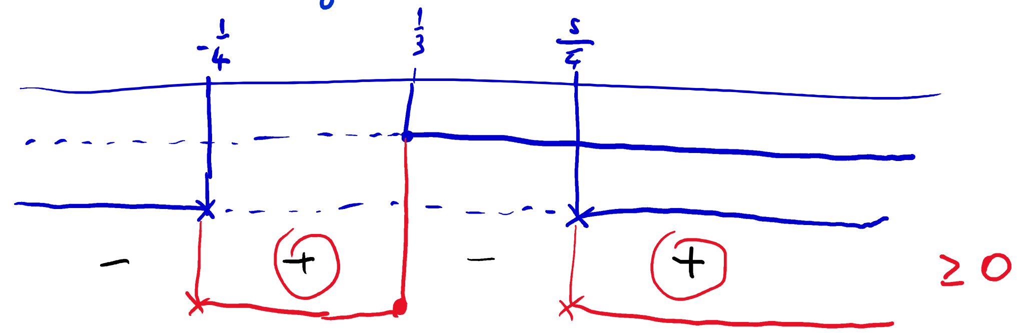
$$x > \frac{5}{4} \qquad \begin{cases} x < \frac{1}{2} \\ -4x > 1 \end{cases}$$

$$\begin{cases} x < \frac{1}{2} \\ x < -\frac{1}{4} \end{cases}$$



$$x < -\frac{1}{4}$$

Studio del segno:



$$-\frac{1}{4} < x \le \frac{1}{3} \quad \vee \quad x > \frac{5}{4}$$

ESERCIZI SUI NUMERI COMPLESSI

ESERCIZIO 1

Sia $z = \dfrac{6+7i}{2-i}$

- Scrivere la forma algebrica di z
- Calcolare $(5i - z)^9$

Soluzione:

- $z = \dfrac{6+7i}{2-i} \cdot \dfrac{2+i}{2+i} = \dfrac{12+14i+6i-7}{4+1} = \dfrac{5+20i}{5}$

  $= 1+4i$

- Sia $w = 5i - z$

  Per il punto precedente:

  $w = 5i - z = 5i - (1+4i) = 5i - 1 - 4i = i - 1 = -1+i$

  $w = -1+i \qquad (x=-1\ ,\ y=1)$

  $|w| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

  $\cos\varphi = -\dfrac{1}{\sqrt{2}}$ e $\sin\varphi = \dfrac{1}{\sqrt{2}} \Rightarrow \varphi = \dfrac{3\pi}{4}$

  $w^9 = (\sqrt{2})^9 \left(\cos\left(\frac{27}{4}\pi\right) + i\sin\left(\frac{27}{4}\pi\right)\right)$

  $= 16\sqrt{2}\left(\cos\left(\frac{3}{4}\pi\right) + i\sin\left(\frac{3}{4}\pi\right)\right)$

$$= 16\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right)$$
$$= -16 + 16\,i$$

- Commenti sul modulo:

- $$(\sqrt{2})^9 = (\sqrt{2})^{8+1} = \underbrace{(\sqrt{2})^8}_{\left(2^{\frac{1}{2}}\right)^8 = 2^{\frac{1}{2}\cdot 8} = 2^4 = 16} \cdot \sqrt{2} = 16\sqrt{2}$$

- $$(\sqrt{2})^7 = (\sqrt{2})^{6+1} = (\sqrt{2})^6 \cdot \sqrt{2} = 8\sqrt{2}$$
  perché:
  $$(\sqrt{2})^6 = \left(2^{\frac{1}{2}}\right)^6 = 2^{\frac{1}{2}\cdot 6} = 2^3 = 8$$

- $$(\sqrt[3]{2})^{11} = (\sqrt[3]{2})^9 \cdot (\sqrt[3]{2})^2 = 8(\sqrt[3]{2})^2 = 8\sqrt[3]{4}$$

- Commento sull'argomento

$$\frac{27}{4}\pi = \left(\frac{24}{4} + \frac{3}{4}\right)\pi = 6\pi + \frac{3}{4}\pi$$

Quindi $\cos\left(\frac{27}{4}\pi\right) = \cos\left(\frac{3}{4}\pi\right) = -\frac{1}{\sqrt{2}}$

$\sin\left(\frac{27}{4}\pi\right) = \sin\left(\frac{3}{4}\pi\right) = \frac{1}{\sqrt{2}}$

ESERCIZIO 2

Sia $z = \dfrac{3i + 11}{2 - 3i}$ e sia $w = z - (3 - \sqrt{3})\,i$

- Scrivere la forma algebrica di z.
- Calcolare w^6
- Scrivere in forma esponenziale le radici quadrate di w

Soluzione:

- $$z = \frac{3i+11}{2-3i} = \frac{(3i+11)(2+3i)}{(2-3i)(2+3i)} = \frac{6i+22+9i^2+33i}{4+9}$$
$$= \frac{6i+22-9+33i}{13} = \frac{13+39i}{13} = 1+3i$$

- $$w = z-(3-\sqrt{3})i$$
$$= 3i+1-(3i-\sqrt{3}i)$$
$$= \cancel{3i}+1-\cancel{3i}+\sqrt{3}i$$
$$= 1+\sqrt{3}i \qquad (x=1, \quad y=\sqrt{3})$$
$$|w| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$$
$$\cos\theta = \frac{1}{2} \qquad \operatorname{sen}\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$
$$w^6 = 2^6\left(\cos\left(6\cdot\frac{\pi}{3}\right)+i\sin\left(6\cdot\frac{\pi}{3}\right)\right)$$
$$= 64\left(\cos(2\pi)+i\sin(2\pi)\right) =$$
$$= 64(1+i\cdot 0)$$
$$= 64$$

- Calcoliamo le radici quadrate di w in forma esponenziale

Abbiamo visto che
$|w| = 2$ e $\arg(w) = \frac{\pi}{3}$
quindi le radici quadrate sono:
$$\sqrt{|w|}\, e^{i\left(\frac{\theta}{2}+k\pi\right)} \quad \text{con } k=0,1$$

cioè

Ricordare:

$w = x+iy$ FORMA ALGEBRICA
$= |z|(\cos\theta+i\sin\theta)$ FORMA TRIG.
$= |z|e^{i\theta}$ FORMA ESPONENZIALE

La formula per le radici n-esime è
$$\sqrt[n]{|w|}\left(\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right)$$
$$= \sqrt[n]{|w|}\, e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)} \qquad k=0,1,\dots,n-1$$

$$\sqrt{2}\, e^{i\left(\frac{\frac{\pi}{3}}{2}+k\pi\right)} \qquad \text{con } k=0,1$$

$$= \sqrt{2}\, e^{i\left(\frac{\pi}{6}+k\pi\right)} = \begin{cases} \sqrt{2}\, e^{i\frac{\pi}{6}} & (k=0) \\ \sqrt{2}\, e^{i\frac{7}{6}\pi} & (k=1) \end{cases}$$

ESERCIZI SUL CALCOLO DI LIMITI CON I POLINOMI DI TAYLOR

ESERCIZIO 1

$$\lim_{x\to 0} \frac{\cos(2x) + \frac{1}{2}\arctan(4x^2) - 1}{2x^2 \ln\left(1+\frac{x}{2}\right) - x^3}$$

Soluzione:

- Numeratore:

$$\cos(2x) \overset{y=2x}{=} \cos y = 1 - \frac{1}{2}y^2 + \frac{1}{24}y^4 + o(y^4)$$

$$= 1 - \frac{1}{2}(2x)^2 + \frac{1}{24}(2x)^4 + o((2x)^4)$$

$$= 1 - \frac{1}{2}\cdot 4x^2 + \frac{1}{24}\cdot 16x^4 + o(x^4)$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 + o(x^4)$$

$$\arctan(4x^2) \overset{y=4x^2}{=} \arctan y$$

$$= y - \frac{1}{3}y^3 + o(y^3)$$

$$= 4x^2 - \frac{1}{3}(4x^2)^3 + o(y^3)$$

$$= 4x^2 - \frac{64}{3}x^6 + o(x^6)$$

$$N(x) = \cancel{1} - 2x^2 + \frac{2}{3}x^4 + o(x^4) + \frac{1}{2}\left(4x^2 - \frac{64}{3}x^6 + o(x^6)\right) \cancel{-1}$$

$$= \cancel{-2x^2} + \frac{2}{3}x^4 + o(x^4) + \cancel{2x^2} \underbrace{- \frac{32}{3}x^6 + o(x^6)}_{= o(x^4)}$$

$$= \frac{2}{3}x^4 + o(x^4)$$

- Denominatore:

$$D(x) = 2x^2 \ln\left(1+\frac{x}{2}\right) - x^3$$

$$\ln\left(1+\frac{x}{2}\right) \overset{y=\frac{x}{2}}{=} \ln(1+y)$$
$$= y - \frac{1}{2}y^2 + o(y^2)$$
$$= \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^2 + o\left(\left(\frac{x}{2}\right)^2\right)$$
$$= \frac{x}{2} - \frac{1}{2}\,\frac{x^2}{4} + o(x^2)$$
$$= \frac{x}{2} - \frac{x^2}{8} + o(x^2)$$

$$D(x) = 2x^2\left(\frac{x}{2} - \frac{x^2}{8} + o(x^2)\right) - x^3$$
$$= x^3 - \frac{x^4}{4} + o(x^4) - x^3$$
$$= -\frac{x^4}{4} + o(x^4)$$

Conclusione:

$$\lim_{x\to 0} \frac{N(x)}{D(x)} = \lim_{x\to 0} \frac{\frac{2}{3}x^4 + o(x^4)}{-\frac{x^4}{4} + o(x^4)} = \lim_{x\to 0} \frac{\frac{2}{3}x^4}{-\frac{x^4}{4}} =$$

$$= \frac{\frac{2}{3}}{-\frac{1}{4}} = \frac{2}{3}\cdot(-4) = -\frac{8}{3}$$

ESERCIZIO 2

$$\lim_{x\to 0} \frac{2x\sin^2 x + \ln(1-2x^3)}{e^{2x^5} - 1}$$

- Denominatore:

$$e^{2x^5} \overset{y=2x^5}{=} e^y$$
$$= 1 + y + o(y)$$
$$= 1 + 2x^5 + o(x^5)$$

Quindi $D(x) = e^{2x^5} - 1 = 1 + 2x^5 + o(x^5) - 1$

$$= 2x^5 + o(x^5)$$

- Numeratore: $N(x) = 2x \sin^2 x + \ln(1-2x^3)$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\sin^2 x = x^2 + \frac{x^6}{36} + o(x^6) - \frac{x^4}{3} + o(x^4) + o(x^6)$$

$$= x^2 - \frac{x^4}{3} + \underbrace{\frac{x^6}{36}}_{=o(x^4)} + o(x^4)$$

$$= x^2 - \frac{x^4}{3} + o(x^4)$$

$$\ln(1-2x^3) \overset{y=-2x^3}{=} \ln(1+y)$$

$$= y - \frac{y^2}{2} + o(y^2)$$

$$= -2x^3 - \frac{(-2x^3)^2}{2} + o((-2x^3)^2)$$

$$= -2x^3 - 2x^6 + o(x^6)$$

$$N(x) = 2x \sin^2 x + \ln(1-2x^3)$$

$$= 2x\left(x^2 - \frac{x^4}{3} + o(x^4)\right) - 2x^3 - 2x^6 + o(x^6)$$

$$= \cancel{2x^3} - \frac{2}{3}x^5 + o(x^5) - \cancel{2x^3} - 2x^6 + o(x^6)$$

$$= -\frac{2}{3}x^5 + o(x^5) - 2x^6 + o(x^6)$$

$$= -\frac{2}{3}x^5 + o(x^5)$$

- Conclusione:

$$\lim_{x\to 0} \frac{N(x)}{D(x)} = \lim_{x\to 0} \frac{-\frac{2}{3}x^5 + o(x^5)}{2x^5 + o(x^5)} = \lim_{x\to 0} \frac{-\frac{2}{3}x^5}{2x^5}$$

$$= \frac{-\frac{2}{3}}{2} = -\frac{2}{3}\cdot\frac{1}{2} = -\frac{1}{3}.$$

ESERCIZI SUGLI INTEGRALI :

ESERCIZIO 1

$$\int \frac{x-2}{x^2+x-2}\,dx$$

Denominatore: x^2+x-2

$\Delta = 1+8=9$

$$x_{1,2} = \frac{-1\pm\sqrt{9}}{2} = \begin{cases} 1 \\ -2 \end{cases}$$

$$x^2+x-2 = (x-1)(x+2)$$

$$\frac{x-2}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$= \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$\begin{aligned} x-2 &= A(x+2)+B(x-1) \\ &= Ax+2A+Bx-B \\ &= x(A+B)+2A-B \end{aligned}$$

$$\begin{cases} A+B=1 \\ 2A-B=-2 \end{cases} \iff \begin{cases} B=1-A \\ 2A-(1-A)=-2 \end{cases} \iff \begin{cases} B=1-A \\ 3A=-1 \end{cases}$$

$$\begin{cases} B=\frac{4}{3} \\ A=-\frac{1}{3} \end{cases}$$

$$\int \frac{x-2}{x^2+x-2}\,dx = \int \frac{-\frac{1}{3}}{x-1}\,dx + \int \frac{\frac{4}{3}}{x+2}\,dx$$

$$= -\frac{1}{3}\int \frac{1}{x-1}\,dx + \frac{4}{3}\int \frac{1}{x+2}\,dx$$

$$= -\frac{1}{3}\ln|x-1| + \frac{4}{3}\ln|x+2| + C$$

ESERCIZIO 2

$$\int \frac{1}{2x^2+x-10}\,dx$$

Denominatore: $2x^2+x-10$

$\Delta = 1+80 = 81 = 9^2$

$$x_{1,2} = \frac{-1\pm 9}{4} = \begin{cases} 2 \\ -\frac{10}{4} = -\frac{5}{2} \end{cases}$$

$$2x^2+x-10 = 2(x-2)\left(x+\frac{5}{2}\right) = (x-2)(2x+5)$$

Cerchiamo A e B t.c.:

$$\frac{1}{2x^2+x-10} = \frac{A}{x-2} + \frac{B}{2x+5}$$

$$= \frac{A(2x+5)+B(x-2)}{(x-2)(2x+5)}$$

$$1 = A(2x+5) + B(x-2)$$
$$= 2Ax + 5A + Bx - 2B$$
$$= x(2A+B) + 5A - 2B$$

Si trova quindi che:

$$\begin{cases} 2A+B = 0 \\ 5A-2B = 1 \end{cases} \Rightarrow \begin{cases} B = -2A \\ 5A+4A = 1 \end{cases} \Rightarrow \begin{cases} B = -\frac{2}{9} \\ A = \frac{1}{9} \end{cases}$$

$$\int \frac{1}{2x^2+x-10}\,dx = \frac{1}{9}\int \frac{1}{x-2}\,dx - \frac{2}{9}\int \frac{1}{2x+5}\,dx$$

$$= \frac{1}{9}\ln|x-2| - \frac{2}{9}\cdot\frac{1}{2}\ln|2x+5| + C$$

$$= \frac{1}{9}\ln|x-2| - \frac{1}{9}\ln|2x+5| + C$$